**Investigating Torque, Moment of Inertia, and Angular Acceleration**

**Objectives**

Upon completion of this lab, students should be able to:

* Describe the parallels between linear and rotational motion by comparing force and torque, mass and moment of inertia, and linear acceleration and angular acceleration.
* Describe how the distribution of mass affects an object’s moment of inertia
* Use a bifilar pendulum to measure the moment of inertia of an object with a complex geometry
* Theoretically calculate the moment of inertia for several simple shapes
* Calculate the angular displacement and the angular acceleration from the angular velocity and make connections to the periodic motion of a pendulum
* Predict how changing the moment of inertia can impact the angular velocity or the amount of torque necessary to carry out certain biomechanical movements

Diver controlling his

rotational motion by changing the moment of inertia of his body.

**Introduction**

In our last experiment, we investigated rotational motion where we used a gyroscope to measure angular velocities and then calculated angular displacement. The next step in our investigation of rotational motion will be to introduce forces that produce rotational motion. In the case of linear motion, we saw that the mass of the object resisted acceleration. We will find that in the case of rotational motion it is not just the amount of mass that resists motion, the resistance also depends on the radial distribution of the mass relative to the axis of rotation. It is insightful to compare linear and rotational motion by examining the form of Newton’s Second Law for each case:

where the sum of the torques, , replaces the sum of the forces, the moment of inertia, , replaces mass, and the angular velocity, , replaces the linear acceleration. In this experiment, we will measure the moment of inertia which is a measure of an objects resistance to change its rotational motion. Mathematically it can be expressed as:

where:

* *mi*is the mass of a small element of the object
* *ri* is the distance of that element from the axis of rotation

The moment of inertia is crucial for characterizing rotational motion in almost every scientific and engineering field, whether you are trying to investigate molecular level movement enabling gene editing enzymes, design a satellite, or explain the vast motion of the universe. In the field of biomechanics, the moment of inertia helps explain how the distribution of mass in the human body affects movement. For instance, athletes, surgeons, and physiotherapists need to understand how the distribution of body mass affects balance and coordination. An athlete with limbs extended will have a higher moment of inertia than one with limbs close to the body, which impacts the speed and ease of rotation. The following illustrations provide examples of the importance of moment of inertia in biomechanics.



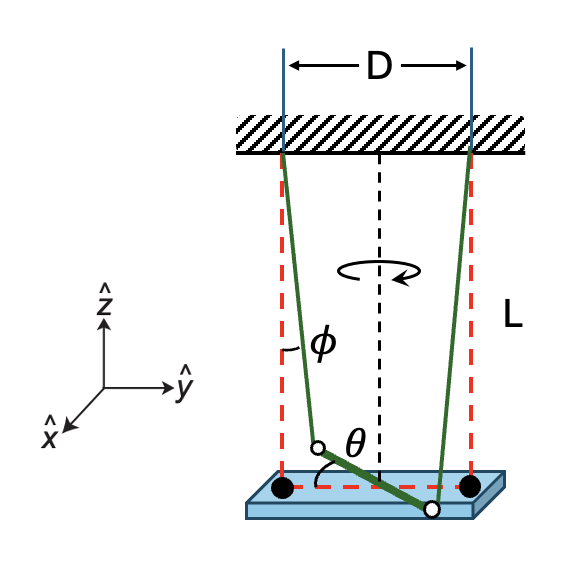
The series of images shows how a sprinter reduces the moment of inertia of his leg as he rotates his hip, thus maximizing the rotational acceleration for a given force.



This straight leg runner will be unable to rotate his hips nearly as fast as a sprinter due to the large moment of inertia of his extended legs. He won’t have much chance in a race with someone who understands the physics of rotational motion and the moment of inertia.

**Pre-Lab Activity (30 minutes)**

1. Locate the heaviest hinged door in your surroundings. Explore the force it takes to accelerate the door when pushing on it with a single finger at different distances from the hinges. Based on your experiment, does the amount of force required to move the door depend linearly on the distance from the hinge (e.g., does pushing in the middle of the door, take about twice the force as pushing on the far edge) or does it feel like the force has a higher order dependence on the distance?

1. Hold your phone in your hand and think about how it balances. Does it feel like the mass of the phone is equally distributed. Try to balance your phone by placing the back of the phone on the tip of your finger (be careful to do this over something soft). Most phones are designed to have the center of mass of the phone very close to the center of the phone.
   1. Do you think the “feel” would be better if the mass was concentrated toward the ends of the phone or the middle? Explain.
   2. How would shifting the components with the most mass toward the middle impact the moment of inertia around the z-axis (i.e., the axis perpendicular to the display of the phone).
2. Estimating the inertial properties of rigid bodies is important in the design of structures which rotate during their motion. Such objects are often irregular shapes such as drones, airplanes, prosthetics, or satellites. While computational methods to calculate the moment of inertia for complex objects are available, it is often preferred to use experimental methods to provide a direct evaluation. One experimental method makes use of the bifilar pendulum. A bifilar pendulum consists of a symmetric object suspended from two parallel wires called filars. These filars allow the body to rotate freely about a given axis. Watch [this NASA video](https://youtu.be/7xQJ2sVQrUA), which uses a bifilar pendulum to measure the inertial properties of a large drone.
   1. Name three of the experimental methods the engineers in this video used to determine the period of oscillation to provide confirmation of the accuracy of their measurements.
3. Consider the simple torsional bifilar pendulum geometry that uses a phone as the bob. When the phone held by the two filars is twisted about the z-axis by a few degrees and let go, as illustrated, it will undergo periodic oscillation. The moment of inertia can be calculated using the relationship shown below, where *T* is the period of oscillation, *m* is the mass of the object, *g* is the acceleration due to gravity, and the other variables are as designated in the figure.   
     
     
     
     
   1. The mathematical analysis of the bifilar pendulum is a little more complex than a traditional simple pendulum, however the basic principle is still similar. Conceptually, explain the origin of the force that produces oscillation in a torsional bifilar pendulum when the phone is displaced from equilibrium by a small rotational movement in the plane perpendicular to *L* and then released. The green lines in the figure indicate the displaced position from equilibrium. (We will walk through this concept in class if you are having trouble visualizing it.)

**Experimental Guide**

A picture containing text, floor

Description automatically generatedIn this experiment a torsional bifilar pendulum is used to measure the moment of inertia of a smartphone along the z-axis by analyzing the period of oscillation. The experimental value of the moment of inertia is then compared to the theoretically calculated value. Finally, the experimental angular velocity data is used to determine both angular displacement and angular acceleration of the smartphone as it undergoes periodic oscillation.

When designing and conducting experimental measurements, researchers have to decide how carefully the experiment needs to be conducted to achieve the specific goals. In some cases, the desired goal can be achieved with a quick measurement and without need to measure each variable with high fidelity. In other experiments, it is essential to pay very close attention to making every measurement precisely. Gaining experience with both scenarios is important. In this experiment, you should be very careful in your measurements. You will be provided some extra guidance to help you through that process.

**Activity 1 – Measuring the Moment of Inertia using a Torsional Bifilar Pendulum (90 minutes)**

Determining the moment of inertia in this activity requires knowledge of several variables associated with the experimental design. Each of the variables will be measured using different tools and each measurement should be made precisely. These measurements include: i) the distance between the parallel filars (measured with a ruler to within ±0.5 mm), ii) the effective length of the filars (measured with a tape measure or meter stick to within ±1 mm), iii) the period of oscillation using the angular velocity (analysis of gyroscope data with the period measured to within ±0.005 s), and iv) the mass of your phone (measured using an electronic balance to within ±0.1 g). In addition to carefully conducting the measurements, the experiment must be thoughtfully designed to assure that the measurements will be physically meaningful.

1. *Optimizing the Experimental Design:* Consider each of the following when constructing your bifilar pendulum. An example experimental design is presented in the figure. Suggestions for the experimental design are included below:
   1. Stability: Using loops to secure the phone will create two points of contact on each side of the phone for each filar and results in a stable geometry. The loops will press against the phone and create well-defined pivot points. This geometry will provide the same motion that would result from the two filars making single contact points.
   2. Precise filar length (*L*): The filars should also have a clear pivot point at the top. This can be achieved by hanging the strings over a well-defined edge. In this case a ruler was secured to the edge of the table so that the edge of the ruler protruded beyond the table edge. (Note that a well-defined pivot point is not generally created if the string is hung over the edge of most tables.) In this case, the distance, *L*, is the straight-line distance from the plane containing these two top pivot points to the parallel plane containing the pivot points on the phone.
   3. Parallel filar spacing (*D*): The filar spacing should be carefully matched at the top pivot points as well as the pivot points where the filars are in contact with the phone. Note that the moment of inertia depends on the square of this spacing.
   4. Level phone: The phone should be level in the x-y plane perpendicular to gravity. Develop a method using a leveling application, the accelerometer with g in phyphox, or the inclination module in phyphox to level the phone.
   5. Small angle oscillation: The equation you are using to model the bifilar pendulum oscillation is only valid for small angles (the derivation of the equations makes the assumption that sin(θ) = θ), so it is best to keep angular displacement to <0.25 rad (15 degrees).
   6. Data collection: Use a timed run of several minutes to collect your data. You might allow yourself 10-30 seconds to start a stable oscillation before you start collecting data. Practice introducing angular displacement of the phone which minimizes any oscillation other than the rotational oscillation around the z-axis. (The maximum angular velocities on the x- and y-axes should be at least an order of magnitude less than the maximum angular velocities on the z-axis.)

Describe any specific changes you made or ideas for improvements of the torsional bifilar pendulum design.

1. *Measure the Angular Velocity and Measure the Bifilar Pendulum Variables*: Collect one high quality data set of the angular velocity for the torsional bifilar pendulum oscillations.
   1. Carefully measure the values of the torsional bifilar pendulum variables and record below:

*D* =

*L* =

*m* =

* 1. Use the gyroscope in phyphox to collect angular velocity data **for at least 2 minutes**. Export the data for analysis and insert a screen shot from phyphox below showing the graphs of angular velocities for all 3 axes.

1. *Measure the Period of Oscillation*: Using a spreadsheet, create a graph of the z-axis angular velocity vs time. Insert a graph of your data below for a time period that illustrates ~5 oscillations.
   1. The next task in the analysis is to measure the period. Be thoughtful in precisely selecting the data points representing the maximum or minimum of oscillations to determine the period. Describe the merits of determining the period from a measurement of the time for a single oscillation versus measuring the time for 5 or 10 oscillations and determining the average period.

* 1. Use the method you judged to be most accurate for calculating the period, to measure the period for a time in the first 20 seconds as well as the last 20 seconds of your experiment.

*Period (T) at early time (t1): t1~ T* =

*Period (T) at later time (t2): t2~ T* =

*Number of oscillations used to determine period =*

* 1. Compare the values for the period determined at early and later times. Our model does not include a time dependence of the period, and we have made an assumption that for small angles, the period will be independent of angle (i.e., independent of amplitude for low amplitude oscillations). Does your observation of the period at early and late times support these assumptions and use of our model?

1. *Calculating the Moment of Inertia around the z-Axis*: Use the equation from the prelab and the values you have measured to calculate the moment of inertia. Show your calculation below.
2. *Comparison of Moment of Inertia of Different Phones*: Compare the moment of inertia you measured for your phone with the moment of inertia measurements of other classmates. Use the table below to record information for several different phones. You can include a measurement of the same model, and for models with different dimensions. If you have measurements for the same model phone, discuss how they compare. Discuss the trends in the moment of inertia for phones with different dimensions. Are the trends consistent with your expectations? Explain.

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| --- | --- | --- | --- |
| Phone Model | Dimensions | Mass (g) | Moment of Inertia (kg m2) |
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**Activity 2 – Comparison of Experimental and Theoretical Moment of Inertia (10 minutes)**

* 1. Chart, diagram, bubble chart

     Description automatically generatedThe moment of inertia for many regular shapes of uniform density can be found in engineering tables, such as that shown in the figure to the right. Calculate the moment of inertia for a rectangular plate of uniform density that has the dimensions of your phone. Show your calculation below for the z-axis of your phone.
     1. Compare the theoretical value to your experimental value. If the difference you measure is accurate, explain what the difference suggests about the mass distribution in your phone.
     2. Think back to your prelab question on how you might distribute the mass in your phone. Was your experimental measurement consistent with your prediction on the optimum design?

**Activity 3 – Analysis and Graphing of Angular Displacement and Angular Velocity (50 minutes)**

The detailed characterization of an object undergoing linear motion was enabled by the relationships between position, velocity, and acceleration. If any of the three variables could be measured, the other two could be calculated. Rotational motion is described by similar relationships that in turn provide the ability to characterize the motion once any one of the variables angular displacement, angular velocity, or angular acceleration is measured. In our case, we use the gyroscope to measure the angular velocity. We can then use the following relationships to determine the angular acceleration and the angular displacement.

1. *Calculate the angular acceleration, α:* The angular acceleration can be calculated from the change in angular velocity for each sampling increment. A model spreadsheet is shown below to guide your analysis. Calculate the angular acceleration for the z-axis angular velocity data you collected previously.

Graphical user interface, text, application

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1. *Calculate the Angular Displacement:* The angular displacement can be determined from the angular velocity using numerical integration. You have done this previously for linear velocity. In that case you defined a coordinate system where the initial position was zero. Determining the zero for a system that is undergoing oscillatory motion, requires more careful consideration. Ideally, *θ* should be set to zero at the equilibrium position, where the net force on the smartphone is zero. The angular displacement will then oscillate around zero. The time when the sum of the torques on the phone will be zero can be identified in the spreadsheet as the time when the angular acceleration is zero. Once that point is identified, the angular displacement at that time can be set to zero and the angular displacement can be calculated using numerical integration for all future times. An example spreadsheet is shown below.

Table

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1. *Graph the Angular Displacement, Angular Velocity, and Angular Acceleration*: You now have measured the angular velocity and calculated the angular acceleration and the angular displacement. Create a graph showing all three variables using a time scale that shows approximately 4 oscillations. Use a secondary axis for the angular acceleration to maximize the visibility of the data.

*Note: If your version of Excel cannot create secondary axes, create two graphs: one with angular velocity and angular acceleration, one with angular velocity and angular displacement.*

* 1. Using the graph as a reference, describe the relationships between angular displacement, angular velocity, and angular acceleration at each of the following locations on the graph:
     1. The angular displacement is a maximum
     2. The angular displacement is a minimum
     3. The angular displacement is zero